Variations in the uncertain parameters affect four of the five open-loop eigenvalues. The eigenvalue sensitivities for these parameters are listed in Table 2 along with the magnitudes of the sensitivity-bound products. The sensitivities are grouped by eigenvalue in the table. Following the procedures of Sec. II, the maximum eigenvalue change for all possible combinations of parameter variations is

$$|\Delta\lambda_{1,2}|_{\max} = 1.63; \qquad |\Delta\lambda_3|_{\max} = 4.47; \qquad |\Delta\lambda_4|_{\max} = 5.10$$

The parameters for scaling are selected by examining Table 2. The parameter M_δ is selected to scale for the change in λ_1 and λ_2. The parameter M_α has the largest sensitivity-bound product for both λ_3 and λ_4. This indicates that M_α can be scaled to accommodate changes in both of these eigenvalues. The initial scaling values are found using Eq. (4): $\Delta M_\delta = 272$; $\Delta M_\alpha = \text{Max}(213, 243) = 243$. Note that because M_α is being used to accommodate changes in two eigenvalues, the maximum of the two initial scaling values is used. By iterating, the final scaled uncertainty values are $\Delta M_\delta = 273$; $\Delta M_\alpha = 259$.

A robust controller is designed using only the scaled uncertainties in M_α and M_δ. Again μ-synthesis was applied with various D-scale orders. D-scales of order two and three were used in the final design. Figure 3 shows the structured singular values of the system, with this autopilot, subject to the original set of uncertainties. Note that the system meets the specifications for robust performance.

IV. Conclusions

A method of reducing the number of modeled uncertainties is presented. The application of this procedure, prior to this autopilot design using μ-synthesis, yielded improvements in the robust performance of autopilots when compared to the direct application of μ-synthesis. Note that the performance comparison is made with respect to the system subject to the complete set of modeled uncertainties. The uncertainty reduction technique presented was applied to three separate missiles, and yielded reductions of 5–50% in the maximum structured singular value in these cases. Because of space considerations, only one of these autopilot designs is presented above. Although the uncertainty reduction technique presented above has been applied only to autopilot design, it is conjectured that it will prove useful in a wider range of applications.

References

[1]Reichert, R. T., "Robust Autopilot Design Using μ-Synthesis," *Proceedings of the 1990 American Control Conference* (San Diego, CA), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1990, pp. 2368–2373.

[2]Wise, K. A., Mears, B. C., and Poolia, K., "Missile Autopilot Design Using H_∞ Optimal Control with μ-Synthesis," *Proceedings of the 1990 American Control Conference* (San Diego, CA), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1990, pp. 2362–2367.

[3]Bibel, J. E., and Stalford, H. L., "An Improved Gain-Stabilized Mu-Controller for a Flexible Missile," AIAA Paper 92-0206, Jan. 1992.

[4]Jackson, P., "Applying μ-Synthesis to Missile Autopilot Design," *Proceedings of the 29th IEEE Conference on Decision and Control* (Honolulu, HI), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1990, pp. 2993–2998.

[5]Enns, D. F., "Rocket Stabilization as a Structured Singular Value Synthesis Design Example," *IEEE Control Systems Magazine*, Vol. 11, No. 4, 1991, pp. 67–73.

[6]Doyle, J. C., "Structured Uncertainty in Control System Design," *Proceedings of the 24th Conference on Decision and Control* (Ft. Lauderdale, FL), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1985, pp. 260–265.

[7]Doyle, J. C., "Analysis of Feedback Systems with Structured Uncertainties," *IEEE Proceedings*, Pt. D, 1982, pp. 242–250.

[8]Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., "State-Space Solutions to Standard H_2 and H_∞ Control Problems," *IEEE Transactions on Automatic Control*, Vol. 34, No. 8, 1988, pp. 831–847.

[9]Frank, P. M., *Introduction to System Sensitivity Theory*, Academic, New York, 1978, p. 217.

Frequency Domain Control of Single Input/Single Output Distributed Parameter Systems

S. M. Yang* and Y. C. Liu†
National Cheng Kung University,
Tainan 701, Taiwan, Republic of China

I. Introduction

ONE of the challenges in structural control is to successfully and confidently apply the control law derived from a discrete, reduced-order model to engineering systems of much higher order. Balas[1] was among the first to show that the spillover associated with discrete system modeling could destabilize a structural control system. There are two types of spillover: observation spillover and control spillover. The former entails the contamination of sensor output through the residual mode dynamics, whereas in the latter, the residual modes are adversely excited by feedback control. Many of the previous studies of the controller design of slewing flexible structures adopted the modal truncation technique in the time domain[2,3] or in the frequency domain.[4,5] Although some systems can be well modeled by a finite number of discretized dominant modes, there are many whose dynamics can only be captured by the distributed parameter model. This Note presents a single input/single output (SISO) controller design for a slewing flexible structure system in distributed parameter model. A stability criterion is developed from the root locus method in the frequency domain, and the criterion is applied to predict the stability of the closed-loop system under a selected control law and a given set of sensor and actuator in noncollocated configurations. It is shown that the control law requires neither modal discretization nor distributed state sensing/estimation; moreover, no functional gain calculation is needed. These advantages are essential for hardware implementation in vibration control application.

II. Stability Analysis

In frequency domain analysis, the open loop transfer function of a SISO system is often written as

$$G_o(\xi, s) = \frac{N_o(\xi, s)}{D_o(s)} \tag{1}$$

where ξ is the spatial coordinate and s the Laplace parameter. Define the transfer function of controller $C(s)$

$$K(s) = k\frac{N_c(s)}{D_c(s)} \tag{2}$$

where $N_c(s)$ and $D_c(s)$ have no common roots, k is a gain parameter, $k > 0$, and the controller is assumed stable. The closed-loop transfer function becomes

$$G_{cl}(\xi, s) = \frac{N_{cl}(\xi, s)}{D_{cl}(\xi, s)} = \frac{D_c(s)N_o(\xi, s)}{D_o(s)D_c(s) - kN_o(\xi, s)N_c(s)} \tag{3}$$

Let C_- denote the open-left-half plane and C_+ the open-right-half plane. The closed-loop eigenvalues will trace the continuous root loci in the complex plane starting from the open-loop poles λ_i and ending at the open-loop zeros z_i as the gain parameter k increases from zero to infinity. The objective is to design a controller such that all closed-loop poles have negative real part thereby staying in C_-. For an undamped flexible structure the root loci will leave the imaginary axis and move into either C_- or C_+ as the control gain

Received May 20, 1994; revision received Jan. 25, 1995; accepted for publication Oct. 30, 1995.

*Associate Professor, Institute of Aeronautics and Astronautics.
†Graduate Research Assistant, Institute of Aeronautics and Astronautics.

k increases from zero. In their study of a string vibration control, Yang and Mote[6] applied the root locus method and showed that the direction of root loci should depend on the real part of their first derivatives with respect to k, which is then evaluated at the open-loop poles. Let $p_i(k)$ denote the root loci originating from λ_i; if the real part of its first derivative at the open-loop pole is negative, then all $p_i(k)$ remain in C_- until the root locus first intersects the imaginary axis at some critical gain k_{cr}. In the case of no intersection between any root locus and the imaginary axis, k_{cr} is infinite. In summary, the stability criterion can be stated as follows: If

$$\mathrm{Re}\left(\left.\frac{\mathrm{d}p_i}{\mathrm{d}k}\right|_{k=0^+}\right) < 0 \tag{4}$$

where

$$\mathrm{Re}\left(\left.\frac{\mathrm{d}p_i}{\mathrm{d}k}\right|_{k=0^+}\right) = \mathrm{Re}\left.\frac{N_o(\xi,\lambda_i)N_c(\lambda_i)}{[\mathrm{d}D_o(s)/\mathrm{d}s]|_{s=j\lambda_i^2}D_c(\lambda_i)}\right|_{k=0^+} \tag{5}$$

then the closed-loop system is asymptotically stable for sufficiently small gain. The transcendental transfer function shown in Eq. (4) is complete in describing the dynamics of a SISO distributed parameter system. Any feasible, implementable control law can be tested for the closed-loop system stability.

III. Controller Design of a Slewing Flexible Beam

The equation of motion for a slewing beam system can be written as

$$v''''(\xi,t) + \ddot{v}(\xi,t) + (r+\xi)\ddot{\theta}(t) = 0 \tag{6a}$$

with boundary conditions

$$J\ddot{\theta}(t) - v''(0,t) + rv'''(0,t) = u \tag{6b}$$

$$v(0,t) = v'(0,t) = v''(1,t) = v'''(1,t) = 0 \tag{6c}$$

where $v(\xi,t)$ is the deflection of the flexible beam, u the control input at the hub, θ the hub attitude, J the rotational inertia of the system, r the hub radius, t time, ξ the spatial coordinate along the beam, $0 \le \xi \le 1$; the prime and dot denote partial differentiation with respect to ξ and t, respectively. All parameters are nondimensionalized. By defining a new coordinate $w(\xi,t)$ as

$$w(\xi,t) = v(\xi,t) + (r+\xi)\theta \tag{7}$$

and taking the Laplace transform of Eq. (6a), a simplified equation can be obtained:

$$\bar{w}''''(\xi,s) + s^2\bar{w}(\xi,s) = 0 \tag{8}$$

with the corresponding boundary conditions in terms of the new coordinate. The overbars denote the Laplace transforms. Let

$$s^2 = -\lambda^4 \tag{9}$$

The solution of Eq. (8) is known to be

$$\bar{w} = A_1\sin(\lambda\xi) + A_2\cos(\lambda\xi) + A_3\sinh(\lambda\xi) + A_4\cosh(\lambda\xi) \tag{10}$$

The open-loop transcendental transfer function from the control torque u to the hub attitude angle θ can be derived analytically as $G_{u\theta} = \bar{\theta}/\bar{u} = N_{u\theta}/D_{u\theta}$, where

$$N_{u\theta} = -(1 + ch_\lambda c_\lambda) \tag{11a}$$

$$D_{u\theta} = \lambda\big[(r^2\lambda^2+1)ch_\lambda s_\lambda + (r^2\lambda^2-1)sh_\lambda c_\lambda + 2r\lambda sh_\lambda s_\lambda + J\lambda^3(1+ch_\lambda c_\lambda)\big] \tag{11b}$$

where $s_\lambda = \sin(\lambda)$, $c_\lambda = \cos(\lambda)$, $sh_\lambda = \sinh(\lambda)$, and $ch_\lambda = \cosh(\lambda)$. Because the rigid body motion and flexible body vibration is coupled as shown in Eq. (7), the transfer function from the hub attitude θ to the tip displacement $v(1)$ is $G_{\theta 1} = N_{\theta 1}/D_{\theta 1}$, where

$$N_{\theta 1} = \lambda(r+1)(1+ch_\lambda c_\lambda) + r\lambda(ch_\lambda + c_\lambda) + (sh_\lambda + s_\lambda) \tag{12a}$$

$$D_{\theta 1} = \lambda(1 + ch_\lambda c_\lambda) \tag{12b}$$

Consequently, the transfer function from the control torque u to the tip displacement $v(1)$ becomes

$$G_{u1} = G_{u\theta}G_{\theta 1} \tag{13}$$

For a selected control law and a given set of sensor and actuator defined by $N_c(s)$ and $D_c(s)$, the closed-loop system stability can be predicted by substituting Eq. (13) into Eq. (5).

Consider the case of colocated feedback of the hub attitude and rate as well as noncollocated feedback of the tip displacement and velocity. The control law is

$$\bar{u} = -(k_0 + k_1 s)\bar{\theta} - (k_p + k_d s)\bar{v}(1) \tag{14}$$

From Eqs. (3) and (5), the root loci derivative can be written as

$$\mathrm{Re}\left(\left.\frac{\mathrm{d}p_i}{\mathrm{d}k}\right|_{k=0^+}\right) = \mathrm{Re}\left[\frac{2jN_{\theta 1}(\lambda_i)}{A_1 ch_{\lambda_i}s_{\lambda_i} A_2 sh_{\lambda_i}c_{\lambda_i} + A_3 sh_{\lambda_i}s_{\lambda_i} + A_4 ch_{\lambda_i}c_{\lambda_i} + A_5}\right] \times \left(k_p + k_d\lambda_i^2 j\right) \tag{15}$$

where λ_i are the open-loop poles of Eq. (11b) and

$$A_1 = 2r^2\lambda_i^2 + 2r\lambda_i^2 - J\lambda_i^4 + k_0 + k_1\lambda_i^2 j$$

$$A_2 = 2r^2\lambda_i^2 + 2r\lambda_i^2 + J\lambda_i^4 - k_0 - k_1\lambda_i^2 j$$

$$A_3 = 2\lambda_i(1+r), \qquad A_4 = 2r^2\lambda_i^3 + 3J\lambda_i^3 - 2k_1\lambda_i j$$

$$A_5 = 3J\lambda_i^3 - 2k_1\lambda_i j$$

The material constants and dimensions of the slewing beam are density $\rho = 2.77$ g/cm^3, Young's modulus $E = 7500$ kg/mm^2, thickness $h = 0.2$ cm, width $w = 3$ cm, length $l = 159$ cm, and hub inertia $J = 840$ g-cm^2. For the design specification of rise time $t_r = 0.5$ s, settling time $t_s = 1$ s, and 1% steady-state error, the collocated control gains are selected as $k_0 = 0.0319$ and $k_1 = 0.0266$. Table 1 lists the root loci derivative of the first five modes of the closed-loop system under the control law of Eq. (14) but with $k_p = 0$. The system is shown stable in the frequency range up to the fifth mode. At higher modes when $\lambda_i \to \infty$,

$$\mathrm{Re}\left(\left.\frac{\mathrm{d}p_i}{\mathrm{d}k}\right|_{k=0^+}\right) \to \frac{-rch_{\lambda_i}\left(2 + c_{\lambda_i}\right)}{J\lambda_i\left(sh_{\lambda_i}c_{\lambda_i} - ch_{\lambda_i}s_{\lambda_i}\right)} \tag{16}$$

The sign of Eq. (16) depends mainly on $\cos(\lambda_i)/[\sin(\lambda_i) - \cos(\lambda_i)]$. It is straightforward to show that the stability criterion in Eq. (5) is satisfied at higher modes as well, provided that λ_i never falls in the region of $[n\pi + (\pi/4), n\pi + (\pi/2)]$ for $n = 1, 2, \ldots$. Although the open-loop poles λ_i, equivalent to the poles of a cantilever beam system, cannot be analytically obtained, they are very similar to periodical solutions, $\lambda_i \to n\pi$, at high-frequency modes. Thus, the slewing beam system under noncollocated feedback is stable. In the case of both velocity and displacement feedback, the additional acquisition of the displacement signal will not affect the stability prediction because the displacement feedback gain k_p appears only in the imaginary part. Note that the stability criterion is applicable to systems with sufficiently small gain. For higher gain, however, the critical value k_{cr} has to be calculated to ensure stability.

To illustrate the advantage of the distributed parameter model over the discrete parameter model, the modal truncation technique is applied to the same slewing beam system. The discrete model admits the first three modes as primary modes and the subsequent two modes as residual modes. The output signals measured, similar

Table 1 Stability criterion test of the distributed parameter model of a slewing flexible beam with noncollocated feedback

Mode number	$\mathrm{Re}\,[(\mathrm{d}p_i/\mathrm{d}k)\vert_{k=0^+}]$
1	−27.405
2	−41.239
3	−35.772
4	−42.973
5	−16.801

Table 2 Closed-loop eigenvalues of the slewing beam in discrete parameter model with three primary modes

Mode	Mode number	Eigenvalue
Primary	1	$-0.0301 \pm 0.301j$
	2	$-0.0233 \pm 10.692j$
	3	$-0.1277 \pm 28.865j$
Residual	4	$-0.0153 \pm 70.155j$
	5	$-1.4552 \pm 327.22j$

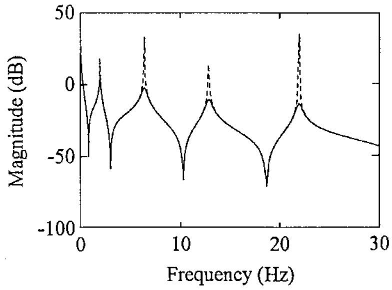

Fig. 1 Bode plot of the slewing flexible beam in distributed parameter model: ——, closed-loop system and - - - -, open-loop system.

to those in Eq. (14), are the hub attitude, angular rate, and tip velocity with the control gains $k_0 = 0.0319$, $k_1 = 0.0266$, and $k_d = 0.016$. Table 2 lists the closed-loop eigenvalues of the discrete model; the loop system is destabilized by the control and observation spillover of residual modes. But this prediction contradicts that in Table 1, which shows that the system is stable. The Bode plot in Fig. 1 shows the distributed parameter model is stable under the same control law and control gains. The example illustrates that inadequate discrete parameter model can lead to incorrect results.

IV. Conclusion

A stability criterion in the frequency domain is developed to predict the closed-loop system stability of a slewing flexible structure in a distributed parameter model. By formulating the open-loop transcendental transfer function, the system dynamics can be fully described so that the spillover problem associated with discrete parameter models can be prevented. The noncollocated feedback controller that uses measurements of the tip displacement and velocity is shown to be effective in vibration suppression; moreover, it is immune from spillover instability. Note that the controller robustness to model uncertainties may require further studies. In the case of discrete system modeling by three primary modes and two residual modes, the closed-loop system under optimal output feedback is predicted to be unstable by eigenvalue calculation. Inadequate discrete parameter model is shown to lead to incorrect results.

References

[1]Balas, M. J., "Trends in Space Structure Control Theory: Fondest Hopes, Wildest Dream," *IEEE Transactions on Automatic Control*, Vol. AC-27, No. 3, 1982, pp. 522–535.

[2]Liu, Y. C., and Yang, S. M., "Vibration Control Experiment of a Slewing Flexible Beam," *Journal of Dynamic Systems, Measurement and Control*, Vol. 117, No. 4, 1995, pp. 432–435.

[3]Liu, Y. C., and Yang, S. M., "Three Simple and Efficient Methods for Vibration Control of Slewing Flexible Structures," *Journal of Dynamic Systems, Measurement and Control*, Vol. 115, No. 4, 1993, pp. 725–730.

[4]Singh, T., and Vadali, S. R., "Robust Time-Optimal Control: Frequency Domain Approach," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 2, 1994, pp. 346–355.

[5]Skarr, S. B., Tang, L., and Yalda-Mooshabad, I., "On-off Attitute Control of Flexible Satellites," *Proceedings of the Guidance, Navigation, and Control Conference*, Vol. 2, AIAA, Washington, DC, 1986, pp. 1222–1228.

[6]Yang, B., and Mote, C. D., "Frequency-Domain Vibration Control of Distributed Gyroscopic Systems," *Journal of Dynamic Systems, Measurement and Control*, Vol. 113, 1991, pp. 18–25.

Optimal Control of a Rigid Body with Dissimilar Actuators

Eric M. Queen*
NASA Langley Research Center, Hampton, Virginia 23681-0001
and
Larry Silverberg†
North Carolina State University, Raleigh, North Carolina 27695-7921

Introduction

SPACECRAFT are frequently controlled by dissimilar actuators such as electrically powered momentum wheels and fuel driven thrusters. In such cases the question arises of how to use the actuators in concert in an optimal manner. Additional difficulties arise when the controls are essentially impulsive, because standard optimization approaches tacitly assume that the controls are bounded or that partial derivatives of the cost with respect to the controls exist.[1] Note that fuel is currently more expensive than electrical power, although this will not be true in every future application, and it will be important to understand the tradeoff.

A geometric approach was used in Refs. 2 and 3 to find the optimal control for a variety of different actuator types. An advantage of this approach is that it easily handles the impulsive solutions associated with minimum fuel controls.[4] This Note applies the geometric approach to a rotating rigid body controlled by dissimilar actuators.

Problem Description

We consider a freely rotating planar rigid body controllable by either or both of two actuators of dissimilar types. The actuators will be referred to as a pair of impulsive thrusters mounted symmetrically about the center of gravity and a torque wheel, although the analysis is applicable to any set of actuators whose cost of operation is proportional to fuel and power, for instance, a cold gas jet and an ion thruster.

The rigid body is rotated about its c.g. through a prescribed angle over a prescribed amount of time. At the end of the maneuver, the body is brought to the origin at rest. The equation governing the planar motion of the rigid body is

$$\theta'' = v_1 + v_2 \tag{1}$$

where θ is the angular position of the rigid body and the nondimensional control inputs are defined by $v_1 = u_1 L T^2/I$ and $v_2 = u_2 T^2/I$. Here the primes denote differentiation with respect to nondimensional time $\tau = t/T$, L is the distance between the thruster and the c.g., u_1 is the force produced by the thrusters, u_2 is the moment produced by the torque wheel, I is the mass moment of inertia of the system, and T is the final time.

Received July 6, 1995; revision received Jan. 23, 1996; accepted for publication Jan. 24, 1996.

*Research Engineer, Vehicle Analysis Branch. Member AIAA.
†Professor, Department of Mechanical and Aerospace Engineering. Member AIAA.